PATH PROPERTIES OF SCHRAMM-LOEWNER EVOLUTION (SLE)

In memory of Oded Schramm

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Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \quad a = \frac{2}{\kappa} > 0,$$

$$U_t = -B_t \quad \text{standard Brownian motion.}$$

- Since $U : [0, \infty) \to \mathbb{R}$ is continuous, there exists simp. conn. $H_t \subset \mathbb{H}$ such that g_t maps H_t conformally onto $\mathbb{H}$ with

$$g_t(z) = z + \frac{a}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

- Does there exist a curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ such that H_t is the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$?
- Is the curve simple? Is $\gamma(0, \infty) \subset \mathbb{H}$?
- What is the Hausdorff dimension of $\gamma(0, t]$?

S. Rohde and O. Scrhamm (2005) Basic properties of SLE, Annals of Math.

EXISTENCE OF THE PATH

$$f_t = g_t^{-1}, \quad \hat{f}_t(z) = f_t(z + U_t).$$

Roughly speaking $\gamma(t) = f_t(U_t) = \hat{f}_t(0)$.

$$\gamma_n(t) = f_t\left(U_t + \frac{i}{n}\right) = \hat{f}_t\left(\frac{i}{n}\right),$$

$$\gamma(t) = \lim_{n\to\infty} \gamma_n(t).$$

Goal: Show limit exists and give bounds on $|\gamma(s) - \gamma(t)|$.

$$|\gamma(t) - \gamma_n(t)| \leq \int_0^{1/n} |\hat f_t'(iy)|\, dy.$$

- If one can show that $|\hat f_t'(iy)| \leq c\, y^{\delta - 1}$ for some $\delta > 0$, then $|\gamma(t) - \gamma_n(t)| \leq O(n^{-\delta})$.
- Given the modulus of continuity of Brownian motion, $U_t - U_s \approx |t-s|^{1/2}$ and distortion estimates for conformal maps, it suffices (up to logarithmic factors) to show that

$$|\hat f_{k/n^2}'(i/n)| \leq c\, n^{1-\delta}, \quad k = 1, 2, \ldots, n^2$$

Let $\epsilon = 1/n$ and $s \leq t \leq s + \epsilon^2$.

$$|\hat{f}_s'(i\epsilon) - \hat{f}_t'(i\epsilon)| = |f_s'(i\epsilon + U_s) - f_t'(i\epsilon + U_t)|$$

$$\leq |f_s'(i\epsilon + U_s) - f_t'(i\epsilon + U_s)| + |f_t'(i\epsilon + U_s) - f_t'(i\epsilon + U_t)|$$

- From the Loewner equation for f we get

$$|f_s'(i\epsilon + U_s) - f_t'(i\epsilon + U_s)| \leq c\,|f_s'(i\epsilon + U_s)|, \quad |s - t| \leq \epsilon^2$$

- Since $|U_t - U_s| \approx \epsilon$, distortion estimates give

$$|f_t'(i\epsilon + U_s) - f_t'(i\epsilon + U_t)| \leq c\,|f_t'(i\epsilon + U_s)|.$$

- Hence $|\hat{f}_s'(i\epsilon)| \approx |\hat{f}_t'(i\epsilon)|$.

Let $\epsilon = 1/n$ and $s \leq t \leq s + \epsilon^2$.

$$|\gamma(s) - \gamma(t)| \leq$$

$$|\gamma(s) - \gamma_n(s)| + |\gamma_n(s) - \gamma_n(t)| + |\gamma_n(t) - \gamma(t)|$$

$$|\gamma_n(s) - \gamma_n(t)| = |f_s(U_s + \epsilon i) - f_t(U_t + \epsilon i)|$$

$$\leq |f_s(U_s + \epsilon i) - f_t(U_s + \epsilon i)| + |f_t(U_s + \epsilon i) - f_t(U_t + \epsilon i)|$$

- $|\gamma_n(s) - \gamma_n(t)|$ can be estimated using mod of cont and disortion estimate.
- Boils down to how well we can estimate $|\hat{f}'_{k/n^2}(i/n)|$ which has the same distribution as $|\hat{f}'_k(i)|$.

$\hat{f}_t'(z)$ has the same distribution as $h_t'(z)$ where h_t follows the reverse Loewner flow:

$$\partial_t h_t(z) = \frac{a}{U_t - h_t(z)}, \qquad h_0(z) = z.$$

$$Z_t = Z_t(z) = X_t + iY_t = h_t(z) - U_t$$

$$M_t(z) = |h_t'(z)|^\lambda \, Y_t^\zeta \, [\sin \arg Z_t]^{-r}$$

$$\lambda = \zeta + \frac{r}{2a} = \left(1 + \frac{1}{2a}\right) - \frac{r^2}{4a}.$$

$M_t(z)$ is a martingale,

$$\mathbf{E}\left[M_t(i)\right] = 1.$$

One hopes that $Y_t(i) \asymp t^{1/2}, [\sin\arg Z_t(i)] \asymp 1$, in which case we can conclude

$$\mathbf{E}\left[|h_t'(i)|^\lambda\right] \asymp t^{-\zeta/2}.$$

- This is correct if $r < 2a + \frac{1}{2}$.
- This estimate is good enough to prove existence of curve for $\kappa \neq 8$.
- For $\kappa = 8$ the existence follows from work of LSW of SLE_κ as limit of Peano curve.
- (Lind, Johansson-L.) $\gamma(t), \epsilon \leq t \leq 1$ is Hölder continuous of order $\alpha < \alpha_*$ but not $\alpha > \alpha_*$ where

$$\alpha_* = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}.$$

$\alpha_* > 0$ unless $\kappa = 8$.

(1/2 times) CONFORMAL RADIUS

Given simp. conn. D and $w \in D$ define

$$\Upsilon_D(w) = \frac{1}{2} f'(0)$$

where $f : \mathbb{D} \to D$ is the conformal transformation with $f(0) = w, f'(0) > 0$. The factor $1/2$ is a convenience so that $\Upsilon_{\mathbb{H}}(i) = 1$.

$$\frac{\Upsilon_D(w)}{2} \leq \mathrm{dist}(w, \partial D) \leq 2\, \Upsilon_D(w).$$

Scaling rule

$$\Upsilon_{f(D)}(f(w)) = |f'(w)|\, \Upsilon_D(w).$$

Given simp. conn. D, distinct boundary points z_1, z_2 and $w \in D$ define

$$\Theta_D(w; z_1, z_2) = \arg F(w), \qquad S_D(w; z_1, z_2) = \sin \arg F(w)$$

where $F : D \to \mathbb{H}$ is a conformal transformation with $F(z_1) = 0, F(z_2) = \infty$.

Let γ be SLE_κ from 0 to ∞ in $\mathbb{H}$ and H_t the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$.

$$\Theta_t = \Theta_{H_t}(w; \gamma(t), \infty).$$

If we reparametrize time ($\tilde{\Theta}_t = \Theta_{\sigma(t)}$) so that $\log \Upsilon_t$ decays linearly, the Loewner equation gives

$$d\tilde{\Theta}_t = (1 - 2a) \cot \tilde{\Theta}_t \, dt + dB_t.$$

If $a \leq 1/4$ ($\kappa \geq 8$), the process never hits zero which implies that the conformal radius goes to zero, i.e., SLE_κ hits points. For $\kappa < 8$, SLE does not hit points.

- For $\kappa < 8$ can determine whether curve goes to right or left of w.

FRACTAL DIMENSION OF $\gamma(0,\infty)$ FOR $\kappa < 8$

- Let d be fractal dimension
- Standard heuristic argument indicates that

$$\mathbf{P}\{\mathrm{dist}[w,\gamma(0,\infty)] \leq \epsilon\} \approx \epsilon^{2-d}.$$

- Similarly we could write

$$\mathbf{P}\{\Upsilon_\infty \leq \epsilon\} \approx \epsilon^{2-d}.$$

Assume there is a function $G_D(w; z_1, z_2)$ such that if γ is SLE_κ from z_1 to z_2 in D,

$$\mathbf{P}\{\Upsilon \leq \epsilon\} \sim G_D(w; z_1, z_2)\, \epsilon^{2-d}, \quad \epsilon \to 0,$$

where

$$\Upsilon = \Upsilon_{D \setminus \gamma}(w).$$

Conformal invariance of SLE implies the scaling rule

$$G_D(w; z_1, z_2) = |F'(w)|^{2-d}\, G_{F(D)}(F(w); F(z_1), F(z_2)).$$

Also,

$$M_t = G_{D \setminus \gamma(0,t]}(w; \gamma(t), z_2)$$

is a local martingale.

Using Itô's formula, we can find that

$$d = 1 + \frac{\kappa}{8}$$

and (up to a multiplicative constant)

$$G_D(w; z_1, z_2) = \Upsilon_D(w)^{d-2}\, S_D(w; z_1, z_2)^{\beta}, \quad \beta = \frac{8}{\kappa} - 1 > 0.$$

Consider $i \in \mathbb{H}$ and let $\Upsilon_t = \Upsilon_{H_t}(i)$ so that $\Upsilon_t \asymp \mathrm{dist}(i, \gamma(0,t])$.

$$M_t = \Upsilon_t^{d-2}\, S_t, \quad S_t = S_{H_t}(i; \gamma(t), \infty).$$

$$\tau_\epsilon = \inf\{t : \Upsilon_t = \epsilon\}.$$

If $\tau_\epsilon = \infty$, $M_\infty = 0$. Would like to say

$$1 = \mathbf{E}[M_{\tau_\epsilon}; \tau_\epsilon < \infty] = \epsilon^{d-2}\, \mathbf{P}\{\tau_\epsilon < \infty\}\, \mathbf{E}[S_{\tau_\epsilon} \mid \tau_\epsilon < \infty]$$

$$\mathbf{E}[S_{\tau_\epsilon} \mid \tau_\epsilon < \infty] \to c_*^{-1}, \quad \mathbf{P}\{\tau_\epsilon < \infty\} \sim c_*\, \epsilon^{2-d}.$$

This can be done (computing c_*) using Girsanov, considering the SDE for S_{τ_ϵ} (function of ϵ) when weighted by the martingale.

It is harder to get second moment estimates

$$\mathbf{P}\{\Upsilon(z) \leq \epsilon, \Upsilon(w) \leq \epsilon\} \asymp \epsilon^{2-d}\,\epsilon^{2-d}\,|z-w|^{d-2},$$

which are needed to prove Hausdorff dimension rigorously. This was done by Beffara.

An alternative proof can be given using the reverse Loewner flow for which the second moment estimates seem to be somewhat easier (but still take work).

NATURAL PARAMETRIZATION

(L-Sheffield, L-Z. Wang)

Parametrize SLE_κ using the "natural" or "fractal" parameterizaton, a d-dimensional param.
D — bounded domain.
$\gamma(t)$ - SLE_κ in D from z_1 to z_2 defined with some param. (e.g., hcap in $\mathbb{H}$). D_t component of $D \setminus \gamma[0,t]$ containing z_2 on boundary.

$$\Theta_t = \text{amount of time in natural param for } \gamma[0,t].$$

$$\mathbf{E}[\Theta_\infty] = \int_D G_D(w; z_1, z_2)\, dA(w).$$

$$\Theta_t + \int_{D_t} G_{D_t}(w; \gamma(t), z_2)\, dA(w)$$

is a martingale.