PATH PROPERTIES OF SCHRAMM-LOEWNER EVOLUTION (SLE)

In memory of Oded Schramm

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Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \quad a = \frac{2}{\kappa} > 0,$$

$$U_t = -B_t \quad \text{standard Brownian motion.}$$

- Since $U : [0, \infty) \to \mathbb{R}$ is continuous, there exists simp. conn. $H_t \subset \mathbb{H}$ such that $g_t$ maps $H_t$ conformally onto $\mathbb{H}$ with

$$g_t(z) = z + \frac{a}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

- Does there exist a curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ such that $H_t$ is the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$?
- Is the curve simple? Is $\gamma(0, \infty) \subset \mathbb{H}$?
- What is the Hausdorff dimension of $\gamma(0, t]$?

**EXISTENCE OF THE PATH**

\[ f_t = g_t^{-1}, \quad \hat{f}_t(z) = f_t(z + U_t). \]

Roughly speaking \( \gamma(t) = f_t(U_t) = \hat{f}_t(0). \)

\[ \gamma_n(t) = f_t \left( U_t + \frac{i}{n} \right) = \hat{f}_t \left( \frac{i}{n} \right), \]

\[ \gamma(t) = \lim_{n \to \infty} \gamma_n(t). \]

Goal: Show limit exists and give bounds on \( |\gamma(s) - \gamma(t)|. \)
\[ |\gamma(t) - \gamma_n(t)| \leq \int_0^{1/n} |\hat{f}'_t(iy)| \, dy. \]

- If one can show that \( |\hat{f}'_t(iy)| \leq c \, y^{\delta-1} \) for some \( \delta > 0 \), then \( |\gamma(t) - \gamma_n(t)| \leq O(n^{-\delta}) \).

- Given the modulus of continuity of Brownian motion, \( U_t - U_s \approx |t - s|^{1/2} \) and distortion estimates for conformal maps, it suffices (up to logarithmic factors) to show that

\[ |\hat{f}'_{k/n^2}(i/n)| \leq c \, n^{1-\delta}, \quad k = 1, 2, \ldots, n^2 \]
Let $\epsilon = 1/n$ and $s \leq t \leq s + \epsilon^2$.

\[
|\hat{f}'_s(i\epsilon) - \hat{f}'_t(i\epsilon)| = |f'_s(i\epsilon + U_s) - f'_t(i\epsilon + U_t)|
\leq |f'_s(i\epsilon + U_s) - f'_t(i\epsilon + U_s)| + |f'_t(i\epsilon + U_s) - f'_t(i\epsilon + U_t)|
\]

- From the Loewner equation for $f$ we get

\[
|f'_s(i\epsilon + U_s) - f'_t(i\epsilon + U_s)| \leq c |f'_s(i\epsilon + U_s)|, \quad |s - t| \leq \epsilon^2
\]

- Since $|U_t - U_s| \approx \epsilon$, distortion estimates give

\[
|f'_t(i\epsilon + U_s) - f'_t(i\epsilon + U_t)| \leq c |f'_t(i\epsilon + U_s)|.
\]

- Hence $|\hat{f}'_s(i\epsilon)| \approx |\hat{f}'_t(i\epsilon)|$. 
Let $\epsilon = 1/n$ and $s \leq t \leq s + \epsilon^2$.

$$|\gamma(s) - \gamma(t)| \leq$$

$$|\gamma(s) - \gamma_n(s)| + |\gamma_n(s) - \gamma_n(t)| + |\gamma_n(t) - \gamma(t)|$$

$$|\gamma_n(s) - \gamma_n(t)| = |f_s(U_s + \epsilon i) - f_t(U_t + \epsilon i)|$$

$$\leq |f_s(U_s + \epsilon i) - f_t(U_s + \epsilon i)| + |f_t(U_s + \epsilon i) - f_t(U_t + \epsilon i)|$$

- $|\gamma_n(s) - \gamma_n(t)|$ can be estimated using mod of cont and distortion estimate.
- Boils down to how well we can estimate $|\hat{f}'_{k/n^2}(i/n)|$ which has the same distribution as $|\hat{f}'_k(i)|$. 

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\( \hat{f}_t'(z) \) has the same distribution as \( h_t'(z) \) where \( h_t \) follows the reverse Loewner flow:

\[
\partial_t h_t(z) = \frac{a}{U_t - h_t(z)}, \quad h_0(z) = z.
\]

\[
Z_t = Z_t(z) = X_t + iY_t = h_t(z) - U_t
\]

\[
M_t(z) = |h_t'(z)|^\lambda Y_t^\zeta [\sin \arg Z_t]^{-r}
\]

\[
\lambda = \zeta + \frac{r}{2a} = \left(1 + \frac{1}{2a}\right) - \frac{r^2}{4a}.
\]

\( M_t(z) \) is a martingale,

\[
\mathbb{E}[M_t(i)] = 1.
\]
One hopes that $Y_t(i) \asymp t^{1/2}$, $[\sin \arg Z_t(i)] \asymp 1$, in which case we can conclude

$$\mathbb{E} \left[ |h'_t(i)|^\lambda \right] \asymp t^{-\zeta/2}.$$ 

- This is correct if $r < 2a + \frac{1}{2}$.
- This estimate is good enough to prove existence of curve for $\kappa \neq 8$.
- For $\kappa = 8$ the existence follows from work of LSW of $SLE_\kappa$ as limit of Peano curve.
- (Lind, Johansson-L.) $\gamma(t), \epsilon \leq t \leq 1$ is Hölder continuous of order $\alpha < \alpha^*$ but not $\alpha > \alpha^*$ where

$$\alpha^* = 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8} + \kappa}.$$ 

$\alpha^* > 0$ unless $\kappa = 8$. 

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Given simp. conn. $D$ and $w \in D$ define

$$\gamma_D(w) = \frac{1}{2} f'(0)$$

where $f : \mathbb{D} \to D$ is the conformal transformation with $f(0) = w$, $f'(0) > 0$. The factor $1/2$ is a convenience so that $\gamma_H(i) = 1$.

$$\frac{\gamma_D(w)}{2} \leq \text{dist}(w, \partial D) \leq 2 \gamma_D(w).$$

Scaling rule

$$\gamma_{f(D)}(f(w)) = |f'(w)| \gamma_D(w).$$
Given simp. conn. $D$, distinct boundary points $z_1, z_2$ and $w \in D$ define

$$\Theta_D(w; z_1, z_2) = \arg F(w), \quad S_D(w; z_1, z_2) = \sin \arg F(w)$$

where $F : D \to \mathbb{H}$ is a conformal transformation with $F(z_1) = 0, F(z_2) = \infty$. 
Let $\gamma$ be $SLE_\kappa$ from 0 to $\infty$ in $\mathbb{H}$ and $H_t$ the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$.

$$\Theta_t = \Theta_{H_t}(w; \gamma(t), \infty).$$

If we reparametrize time ($\tilde{\Theta}_t = \Theta_{\sigma(t)}$) so that $\log \gamma_t$ decays linearly, the Loewner equation gives

$$d\tilde{\Theta}_t = (1 - 2a) \cot \tilde{\Theta}_t \, dt + dB_t.$$

If $a \leq 1/4$ ($\kappa \geq 8$), the process never hits zero which implies that the conformal radius goes to zero, i.e., $SLE_\kappa$ hits points. For $\kappa < 8$, $SLE$ does not hit points.

- For $\kappa < 8$ can determine whether curve goes to right or left of $w$. 

FRACTAL DIMENSION OF $\gamma(0, \infty)$ FOR $\kappa < 8$

- Let $d$ be fractal dimension
- Standard heuristic argument indicates that
  \[ P\{\text{dist}[w, \gamma(0, \infty)] \leq \epsilon\} \approx \epsilon^{2-d}. \]
- Similarly we could write
  \[ P\{\Upsilon_{\infty} \leq \epsilon\} \approx \epsilon^{2-d}. \]
Assume there is a function $G_D(w; z_1, z_2)$ such that if $\gamma$ is $SLE_\kappa$ from $z_1$ to $z_2$ in $D$,

$$P\{\Upsilon \leq \epsilon\} \sim G_D(w; z_1, z_2) \epsilon^{2-d}, \quad \epsilon \to 0,$$

where

$$\Upsilon = \Upsilon_{D \setminus \gamma}(w).$$

Conformal invariance of $SLE$ implies the scaling rule

$$G_D(w; z_1, z_2) = |F'(w)|^{2-d} G_{F(D)}(F(w); F(z_1), F(z_2)).$$

Also,

$$M_t = G_{D \setminus \gamma(0,t]}(w; \gamma(t), z_2)$$

is a local martingale.
Using Itô’s formula, we can find that

\[ d = 1 + \frac{\kappa}{8} \]

and (up to a multiplicative constant)

\[ G_D(w; z_1, z_2) = \gamma_D(w)^{d-2} S_D(w; z_1, z_2)^\beta, \quad \beta = \frac{8}{\kappa} - 1 > 0. \]
Consider $i \in \mathbb{H}$ and let $\gamma_t = \gamma_{H_t}(i)$ so that $\gamma_t \approx \text{dist}(i, \gamma(0, t))$.

$$M_t = \gamma_t^{d-2} S_t, \quad S_t = S_{H_t}(i; \gamma(t), \infty).$$

$$\tau_\epsilon = \inf\{t : \gamma_t = \epsilon\}.$$ If $\tau_\epsilon = \infty$, $M_\infty = 0$. Would like to say

$$1 = \mathbb{E}[M_{\tau_\epsilon}; \tau_\epsilon < \infty] = \epsilon^{d-2} \mathbb{P}\{\tau_\epsilon < \infty\} \mathbb{E}[S_{\tau_\epsilon}; \tau_\epsilon < \infty]$$

$$\mathbb{E}[S_{\tau_\epsilon}; \tau_\epsilon < \infty] \rightarrow c_*^{-1}, \quad \mathbb{P}\{\tau_\epsilon < \infty\} \sim c_* \epsilon^{2-d}.$$ This can be done (computing $c_*$) using Girsanov, considering the SDE for $S_{\tau_\epsilon}$ (function of $\epsilon$) when weighted by the martingale.
It is harder to get second moment estimates

\[ P\{\Upsilon(z) \leq \epsilon, \Upsilon(w) \leq \epsilon\} \asymp \epsilon^{2-d} \epsilon^{2-d} |z - w|^{d-2}, \]

which are needed to prove Hausdorff dimension rigorously. This was done by Beffara. An alternative proof can be given using the reverse Loewner flow for which the second moment estimates seem to be somewhat easier (but still take work).
NATURAL PARAMETRIZATION
(L-Sheffield, L-Z. Wang)

Parametrize $SLE_\kappa$ using the “natural” or “fractal” parametrization, a $d$-dimensional param. $D$ — bounded domain.
$\gamma(t)$ - $SLE_\kappa$ in $D$ from $z_1$ to $z_2$ defined with some param. (e.g., hcap in $\mathbb{H}$). $D_t$ component of $D \setminus \gamma[0, t]$ containing $z_2$ on boundary.

$\Theta_t = \text{amount of time in natural param for } \gamma[0, t].$

$$E[\Theta_\infty] = \int_D G_D(w; z_1, z_2) \, dA(w).$$

$$\Theta_t + \int_{D_t} G_{D_t}(w; \gamma(t), z_2) \, dA(w)$$

is a martingale.