Rigidity of Circle Packings

Ken Stephenson

University of Tennessee

Oded Schramm Memorial, 8/2009
Circle Packing – Background
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An associated circle packing $P$
Existence and Uniqueness

Theorem: Given any triangulation $K$ of a topological sphere, there exists a univalent circle packing $P_K$ of the Riemann sphere having the combinatorics of $K$. Moreover, $P_K$ is unique up to Möbius transformations and inversion.

Theorem: Given a triangulation $K$ of any oriented topological surface $S$, there exists a conformal structure on $S$ and a univalent circle packing $P_K$ in its intrinsic metric, so that $P_K$ "fills" $S$. Moreover, the conformal structure is unique and $P_K$ is unique up to its conformal automorphisms.

Upshot: Circle packings endow combinatorial situations with geometry. • Local rigidity • Global flexibility • and this is a particularly familiar geometry — it’s conformal!

Oded’s frequent collaborator, Zheng-Xu He, will say more about this in the next talk.
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Thurston’s Conjecture, 1985

Conjecture: Under refinement, the discrete conformal maps \( f : P \rightarrow K \) converge uniformly on compacta to the classical conformal map \( F : D \rightarrow \Omega \).

Rodin and Sullivan proved the conjecture, which has been vastly extended—under refinement, objects in the discrete world of circle packing invariably converge to their classical counterparts.
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Rigidity

Claim:
If $P$ and $P'$ are two circle packings of the sphere sharing the combinatorics of $K$, then they are Möbius images of one another.

The crucial tool?
Two circles can intersect in at most two points.
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The setup, I
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P

P'
The setup, I
The setup, II

Put $\infty$ in the chosen interstice and project both packings to the plane to get these juxtaposed configurations:
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Put $\infty$ in the chosen interstice and project both packings to the plane to get these juxtaposed configurations: Scale $P$ away from $a$ to put the packings in general position:
The “elements” of $P$ and $P'$
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**Elements:**

circle elements
The "elements" of $P$ and $P'$

Elements:
- circle elements
- interslice elements
The “elements” of $P$ and $P'$

**Elements:**
- circle elements
- \( \bigcup \) interstice elements

\[ = E \]
The “elements” of $P$ and $P'$

Elements:

circle elements $\bigcup$ interstice elements

$= E$
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Elements:
- circle elements \( \bigcup \)
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\[ = E \]

Likewise for $P'$

\[ E \leftrightarrow E' \]
The “elements” of $P$ and $P'$

Elements:
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Comparison via “Fixed point index”

Definition:
Given simple closed curves $\gamma$ and $\sigma$ and an orientation preserving, fixed-point-free homeomorphism $f: \gamma \rightarrow \sigma$, the fixed point index $\eta(f; \gamma)$ is the winding number of $g(z) = f(z) - z$ about $\gamma$. 

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Definition: Given simple closed curves $\gamma$ and $\sigma$ and an orientation preserving, fixed-point-free homeomorphism $f : \gamma \xrightarrow{fpf} \sigma$, the fixed point index $\eta(f; \gamma)$ is the winding number of $g(z) = f(z) - z$ about $\gamma$. 
Compatibility

If $\gamma$ and $\sigma$ are both circles then for every $f$: $\gamma \xrightarrow{fpf} \sigma$, $\eta(f; \gamma) \geq 0$.

If $\gamma = \langle a, b, c \rangle$ and $\sigma = \langle a', b', c' \rangle$, then there exists $f$: $\gamma \xrightarrow{fpf} \sigma$ with $\eta(f; \gamma) \geq 0$. 
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If \( \gamma = \langle a, b, c \rangle \) and \( \sigma = \langle a', b', c' \rangle \), then there exists \( f : \gamma \xrightarrow{\text{ipf}} \sigma \) with

\[ \eta(f; \gamma) \geq 0. \]
The Proof

\[ \forall \text{ interstice element } e_j \in E \text{ choose } f_j : e_j \xrightarrow{fpf} e_j' \text{ so that } \eta(f_j; e_j) \geq 0. \]

\[ \forall \text{ circle element } e_k, \text{ define } f_k : e_k \xrightarrow{fpf} e_k' \text{ to agree with the maps of neighboring interstices.} \]

\[ \text{The element maps induce a homeomorphism } F : \Gamma \xrightarrow{fpf} \Sigma \text{ between the outer boundaries of our two configurations.} \]

\[ \text{Taking account of cancellations on interior segments, } \eta(F; \Gamma) = \sum_{e_j \in E} \eta(f_j; e_j). \]

\[ \text{In particular, } \eta(F; \Gamma) \geq 0. \]
The Proof

• ∀ interstice element $e_j \in E$ choose $f_j : e_j \mapsto e'_j$ so that $\eta(f_j; e_j) \geq 0$. 

• ∀ circle element $e_k$, define $f_k : e_k \mapsto e'_k$ to agree with the maps of neighboring interstices.

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The Proof

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• The element maps induce a homeomorphism \( F : \Gamma \stackrel{i^pf}{\rightarrow} \Sigma \) between the outer boundaries of our two configurations.

• Taking account of cancellations on interior segments,

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\eta(F; \Gamma) = \sum_{e_j \in E} \eta(f_j; e_j).
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• In particular, \( \eta(F; \Gamma) \geq 0 \)
but ... 

\[ F : \Gamma \xrightarrow{fpf} \Sigma \text{ and } \eta(F; \Gamma) \geq 0 \]
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but ...

\[ F : \Gamma \xrightarrow{fpf} \Sigma \quad \text{and} \quad \eta(F; \Gamma) \geq 0 \]

By observation, \[ \eta(F; \Gamma) = -1 \]
The other bookend
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Theorem:
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Theorem: [Schramm/He] The KAT Theorem on circle packings of the sphere implies the Riemann Mapping Theorem for plane domains.
Existence

Theorem: Given any Jordan region $\Omega$, there exists a univalent circle packing with heptagonal combinatorics which fills $\Omega$. Moreover, the packing is unique subject to standard normalization.
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ODED

Schramm

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Thanks

“Packing two-dimensional bodies ...”,
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